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principal projection property of Riesz space

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Abstract:
This research considers principal projection property of Riesz space, which is the basis of positive operators and their applications. And provides important theorems concerns useful properties of projection elements. Also, describes lattice operations of order bounded operators and retract of Riesz space.
Keywords: projection, ideal, Riesz space, Dedekind complete.

خاصية الإسقاط الرئيسية لفضاء ريـــــس

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الملخص:
يختص هذا البحث بدراسة خاصية الإسقاط الرئيسية لفضاء ريس التي تعتبر خاصية أساسية للمؤثرات الموجبة وتطبيقاتها، حيث أنه تم عرض بعض النظريات المهمة التي تدرس خواص عناصر الإسقاط وكذلك قدم هذا دراسة العمليات الشبكية للمؤثرات المحدودة ترتيبياً وخاصية التراجع لفضاء ريس.
الكلمات المفتاحية: خاصية الإسقاط، المثالية، فضاء ريس، فضاء ديديكند التام.

Introduction:

In this research we study principal projection property of Riesz space. In the beginning we have to know that, if every element in Riesz space is projection element, then Riesz space is said to be has principal projection property. Also, we show some theorems concerns useful properties of projection elements. Then we describe the lattice operations of order bounded operators, from a space which has principal projection property into Dedekind complete space. This research deals with retracts of Riesz spaces. Many researchers studied this topic some of them, (F.Riesz, 1930,143) proven that in a Dedekind complete Riesz space every band is a projection band, (Aliprantis, 1985,33) find a new formula for the order projection onto the band generated by a positive operator, depending on the results of (Aliprantis, 1980,245), and (M. Volodymyr etc, 2020, 291) proven the existence of lateral band projection and principal projection property implies the intersection property.

Definition 1 (Aliprntis, 1985, 1)
In any ordered vector space X any element x is called positive whenever $x \geq 0$.
The set of all positive elements of X will be denoted by X^+.

Definition 2 (Helmut, 1974, 56)
A subset U of a Riesz space X is called **solid** if $x \in U, y \in X$, and $|x| \leq |y|$ implies $y \in U$. A solid vector subspace U of X is called an ideal of X.

Definition 3 (John, 1990, 191)
An ideal U of a Riesz space X is called a band if $A \subset U$ and $\sup A = x \in X$ implies $x \in U$.

If U is a nonempty subset of a Riesz space X. Then the ideal generated by U is the smallest ideal that contains U. We can write it as;

$$\left\{ x \in X : \exists x_1, \ldots, x_n \in U \;\; and \;\; \lambda_1, \ldots, \lambda_n \in R^+ \;\; with \; |x| \leq \sum_{i=1}^{n} \lambda_i |x_i| \right\} \;\ldots\ldots\; (1).$$

The ideal generated by an element x will be denoted by U_x. From equation (1) we can define U_x as:

$$U_x = \{ y \in X : \exists \; \lambda > 0 \; with \; |y| \leq \lambda |x| \}.$$

Every ideal of the form U_x is called as a **principal ideal.** By the same way, the band generated by U is the smallest band that contains U. Such a band always exists (since it is the intersection of the family of all bands that contain U, and X is one of them).
It is obvious that, the band generated by U agrees with the band generated by the ideal generated by U.

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Example1. (Helmut, 1974, 57)
If X is any Riesz space. And $x \in X^+$ then the order interval $[-x, x]$ is solid, and the vector subspace $\bigcup_1^\infty n[-x, x]$ is the principal ideal X_x

Clearly, if U is a directed $(\le)$ subset of X^+ then $\bigcup_1^\infty \{n[-x, x] : n \in \mathrm{N}, x \in U\}$ is the ideal generated by U. Any union of solid sets is solid, and $V \subset X$ is solid if $V = \bigcup \{ [-|x|, |x|] : x \in V \}$.

The following theorem describes the band generated by an ideal.

Theorem 1. Let U be an ideal of a Riesz space X. Then the band generated by U is precisely

$$\{x \in X : \exists \{x_\alpha\} \subseteq U^+ \; with \; 0 \le x_\alpha \uparrow |x|\},$$

In general, every ideal is order dense in the band it generates. Moreover, the band V_x generated by a single element x satisfies

$$V_x = \{y \in X : |y| \wedge n\,|x| \uparrow |y|\}.$$

Proof. Let $V = \{x \in X : \exists \{x_\alpha\} \subseteq U^+ \; with \; 0 \le x_\alpha \uparrow |x|\}$. Clearly, every band containing U must contain V. Thus, to establish our result it is enough to show that V is a band.
To this end, let $x, y \in V$. Pick two nets $\{x_\alpha\} \subseteq U^+$ and $\{y_\beta\} \subseteq U^+$ with $0 \le x_\alpha \uparrow |x|$ and $0 \le y_\alpha \uparrow |y|$. From

$$|x + y| \wedge (x_\alpha + y_\beta) \uparrow |x + y| \wedge (|x| + |y|) = |x + y|$$

And

$$|\lambda|\, x_\alpha \uparrow |\lambda\, x|$$

It is obvious that, V is vector subspace. Also, if $|z| \le |x|$ holds, then from

$$\{|z| \wedge x_\alpha\} \subseteq U \;\text{ and }\; 0 \le |z| \wedge x_\alpha \uparrow |z| \wedge |x| = |z|,$$

It follows that $z \in V$. Hence, V is an ideal.
Finally, to see that V is a band, let $\{x_\alpha\} \subseteq V$ satisfy $0 \le x_\alpha \uparrow |x|$. Let $D = \{ y \in U : \exists \; some \; \alpha \; with \; 0 \le y \le x_\alpha \}$.
Then $D \subseteq U^+$ and $D \uparrow x$ hold. Therefore, $x \in V$, and so V is a band.

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To establish the formula for V_x, let $y \in V_x$.

By the above there exists a net $\{x_\alpha\} \subseteq U_x$ with $0 \le x_\alpha \uparrow |y|$. Now given α there exists some n with $x_\alpha \le n|x|$, and so $x_\alpha \le |y| \wedge n|x| \le |y|$ holds. This easily implies $|y| \wedge n|x| \uparrow |y|$

Definition 4 (Aliprntis, 1985, 5)
disjoint complement U^d is defined by $U^d = \{x \in X : x \perp y \; for\, all \; y \in U\}$.
Note that $U \cap U^d = \{0\}$, and $U^{dd} = (U^d)^d$.
It is an obvious that is always a band.

Definition 5 (Aliprntis, 1985, 33)
an operator $F : X \to Y$ between two vector space is said to be a **Projection** whenever $F = F^2$.

The next theorem describes the useful condition which an ideal is necessarily a band.

Theorem 2. If U and V are two ideals in a Riesz space X such that $X = U \oplus V$. Then U and V are both bands satisfying $U = V^d$ and $V = U^d$ (and hence $U = U^{dd}$ and $V = V^{dd}$ both hold).
Proof. Note first that for each $u \in U$ and $v \in V$ we have
$$|u| \wedge |v| \in U \cap V = \{0\},$$
And so $U \perp V$. In particular, $U \subseteq V^d$.
On the other hand , if $x \in V^d$, then write $x = u + v$ with $u \in U$ and $v \in V$, and note that $v = x - u \in V \cap V^d = \{0\}$ implies $x = u \in U$. Thus, $V^d \subseteq U$, and so $U = V^d$ holds. This shows that U is a band. By the symmetry of the situation $V = U^d$ also hold.

Definition6 (Aliprntis, 1985,33)
A band V in a Riesz space X is called a **projection band** whenever $X = V \oplus V^d$ **.**

The next theorem describes the ideals that are projection bands.

Theorem 3. If V is an Ideal in a Riesz space *X*, then the following statements are equivalent:

1. V is a projection band, i.e., $X = V \oplus V^d$ holds.
2. For each $x \in X^+$ the supremum of the set $V^+ \cap [0, x]$ exists in *X* and belongs to *V*.
3. There exists an ideal U of X such that $X = V \oplus U$ holds.

Proof (1) $\Rightarrow$(2)

Let $x \in X^+$. Choose the (unique) elements $0 \le y \in V$

And $0 \le z \in V^d$ with $x = y + z$. If $u \in V^+$ satisfies $u \le x = y + z$, then it follows from $0 \le (u-y)^+ \le z \in V^d$ and $(u-y)^+ \in V$ that $(u-y)^+ = 0$. Thus, $u \le y$, and so y is an upper bound of the set $V^+ \cap [0, x]$. Since $y \in V \cap [0, x]$, we see that $y = \sup\{u \in V^+ : u \le x\} = \sup V \cap [0, x]$ holds in X.

(2) $\Rightarrow$ (3) Fix some $x \in X^+$, and let $u = \sup V \cap [0, x]$. Clearly, *u* belongs to *V*. Put $y = x - u \ge 0$. If $0 \le w \in V$, then $0 \le y \wedge w \in V$, and moreover from $0 \le u + y \wedge w \in V$ and $u + y \wedge w = (u+y) \wedge (u+w) = x \wedge (u+w) \le x$, it follows that

$u + y \wedge w \le u$. Hence, $y \wedge w = 0$ holds, and so $y \in V^d$. From $x = u + y$ we see that $X = V \oplus V^d$, and therefore (3) holds with $U = V^d$.

(3) $\Rightarrow$(1) This follows from Theorem 2.

Theorem 4 (Aliprntis, 1985, 33). If *V* is a band in a Dedekind complete Riesz space *X*, then $X = V \oplus V^d$ holds.

If F a positive operator defined on a Riesz space and F is also a projection, then F will be named to as **a positive Projection**.

Now let V be a projection band in a Riesz space X. Thus, $X = V \oplus V^d$ holds, and so every element $x \in X$ has a unique decomposition $x = x_1 + x_2$, where $x_1 \in V$ and $x_2 \in V^d$. Then it is easy to see that a projection $F_V : X \to X$ is defined by:

$$F_V(x) = x_1.$$

Clearly, F_V is a positive projection. Any projection of the form F_V is named an **order projection** or (**a band projection**). Thus, the order projection is associated with the projection bands in a one-to-one fashion.

Theorem 5. If V is a projection band in a Riesz space X, then

$$F_V(x) = \sup\{y \in V : 0 \le y \le x\}$$

Holds for all $x \in X^+$.

Proof. Let $x \in X^+$. Then from proof of (Theorem 3) $u = \sup\{y \in V : 0 \le y \le x\}$ exists and belongs to V. Assume that $u = F_V(x)$. Write $x = x_1 + x_2$ with $0 \le x_1 \in V$ and $0 \le x_2 \in V^d$, and note that $0 \le x_1 \le x$ implies $0 \le x_1 \le u$. Thus, $0 \le u - x_1 \le x - x_1 = x_2$, and hence $u - x_1 \in V^d$ since $u - x_1 \in V$ and $V \cap V^d = \{0\}$, we see that $u = x_1$, as assumed.

The next theorem describes the basic properties of order projections.

Theorem 6. If U and V are projection bands in Riesz space X, then U^d, $U \cap V$, and $U + V$ are likewise projection bands. Moreover, they satisfy

1. $F_{U^d} = I - F_U$;
2. $F_{U \cap V} = F_U F_V = F_V F_U$; and
3. $F_{U+V} = F_U + F_V - F_U F_V$.

Proof. (1) From $X = U \oplus U^d$ it follows that $U^{dd} = U$ holds (see Theorem 1), and so U^d is a projection band. The identity $F_{U^d} = I - F_U$ should be obvious.

(2) To see that $U \cap V$ is a projection band note that the identity $V \cap [0, x] = [0, F_V x]$ implies $U \cap V \cap [0, x] = U \cap [0, F_V x]$ for each $x \in X^+$. Thus,

$F_U F_V x = \sup\ U \cap V \cap [0, x] = \sup U \cap [0, F_V x]$

Holds for each $x \in X^+$, which (by theorem 3) shows that $U \cap V$ is a projection band and that $F_{U \cap V} = F_U F_V$ holds. Similarly, $F_{U \cap V} = F_U F_V$.

(3) Assume at the beginning that the two projection bands U and V satisfy $U \perp V$. If $x \in X^+$. And $0 \le u + v \in U + V$ satisfy $u + v \le x$, then $u \in U \cap [0, x]$ and $v \in V \cap [0, x]$, and so $u + v \le F_U x + F_V x \in U + V$ holds.

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This shows that

$$\sup(U+V)\cap[0,x]=F_U x+F_V x\in U+V ,$$

From proof of (Theorem 3) we have the ideal $U+V$ is a projection band. Also, $F_{U+V}=F_U+F_V$ holds.

Now the general case can be established by observing that $U+V=U\cap V^d+V$. Also, we have

$$F_{U+V}=F_{U\cap V^d+V}=F_{U\cap V^d}+F_V=F_U F_{V^d}+F_V$$
$$=F_U(I-F_V)+F_B=F_U-F_U F_V+F_V=F_U+F_V-F_{U\cap V}\text{, as}$$

required.

An immediate consequence of statement (2) of the preceding theorem is that two arbitrary order projection mutually commute.

The next theorem describes the useful comparison property of order projection.

Theorem 7. If U and V are projection bands in Riesz space X, then the following statements are equivalent:

1. $U\subseteq V$
2. $F_U F_V=F_V F_U=F_U$; and
3. $F_U\le F_V$.

Proof. (1)$\Rightarrow$(2) Let $U\subseteq V$. Then from Theorem 6 it follows that

$$F_U F_V=F_V F_U=F_{U\cap V}=F_U$$

(2)$\Rightarrow$(3) For each $0\le x$ we have $F_U x=F_V F_U x\le F_V x$, and so $F_U\le F_V$ holds.

(3)$\Rightarrow$(1) If $0\le x\in U$, then it follows from

$$0\le x=F_U x\le F_V x\in V$$

That $x\in V$. Therefore, $U\subseteq V$ holds, as required.

Definition 7 (Aliprntis, 1985, 33)

the band V_x generated by a single element x satisfies

$$V_x=\{y\in X:|y|\wedge n|x|\uparrow|y|\}.$$

Definition 8 (Aliprntis, 1985, 35)

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An element x in a Riesz space is said to be a **projection element** whenever the band V_x generated by x ($V_x = \{ y : |y| \wedge n|x| \uparrow |y| \}$ is a projection band.

Definition 9 (Aliprntis, 1985, 35)
If every element in a Riesz space is projection element, then the Riesz space is said to have the **principal projection property**

Definition 10 (Helmut, 1974, 63)
The band V_x of X generated by a single element x is called a **principal band** of X. If each principal band of X is a projection band, X is said to have the **principal projection property**.

For a projection element x we shall write F_x for the order projection onto the band V_x.

Theorem 8. An element x in a Riesz space is a projection element if and only if $\sup\{ y \wedge n\,|x| \}$ exists for each $y \geq 0$. In this case

$F_x(y) = \sup\{ y \wedge n\,|x| \}$ holds for all $y \geq 0$.

Proof. Let $y \geq 0$. We assume at that the two sets $V_x \cap [0, y]$ and $\{ y \wedge n\,|x| : n \geq 1\}$ have the same upper bounds. To see this, note first that $\{ y \wedge n\,|x| \} \subseteq V_x \cap [0, y]$ holds. Now let $y \wedge n\,|x| \leq u$ for all n. If $z \in V_x \cap [0, y]$, then by Theorem 1we have $z \wedge n\,|x| \uparrow z$. In view of $z \wedge n\,|x| \leq y \wedge n\,|x| \leq u$, we see that $z \leq u$, and so the two sets have the same upper bounds.
From the theorem 3 and theorem 5 it follows immediately that in a $\sigma -$ Dedekind complete Riesz space every principal band is a projection band. If $x, y \geq 0$ are projection element in a Riesz space, then note that the formulas of Theorem 6 take the form

$F_{x \wedge y} = F_x F_y = F_y F_x$ and $F_{x+y} = F_x + F_y - F_{x \wedge y}$.

An element $e > 0$ in a Riesz space X is said to be a **weak order unit** whenever the band generated by e satisfies $V_e = X$ (or, equivalently, whenever for each $x \in X^+$ we have $x \wedge ne \uparrow x$). Now, every element

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$0 < x \in X$ is a weak order unit in the band it generates. Also, note that an element $e > 0$ is a weak order unit if and only if $x \perp e$ implies $x = 0$.
The next theorem describes the following useful properties of Projection element.

Theorem 9. If u, v and w are projection elements in a Riesz space X . Then the following statements hold:

1. If u, v and w satisfying $0 \le w \le v \le u$, then for each $x \in X$ we have $(F_u - F_v)x \perp (F_v - F_w)x$.
2. If $0 \le u_\alpha \uparrow u$ holds in X with u and all the u_α projection elements, then $F_{u_\alpha}(x) \uparrow F_u(x)$ holds for each $x \in X^+$.

Proof. (1) Froom proof of Theorem 7 we have $F_w \le F_v \le F_u$, and so

$$\begin{aligned} 0 &\le |(F_u - F_v)x| \wedge |(F_v - F_w)x| \\ &\le (F_u - F_v)|x| \wedge (F_v - F_w)|x| \\ &\le [F_u|x| - F_v(F_u|x|)] \wedge F_v(F_u|x|) = 0. \end{aligned}$$

(2) Let $x \in X^+$. Clearly, $F_{u_\alpha}(x) \uparrow F_u(x)$. Thus, $F_u(x)$ is an upper bound for $\{F_{u_\alpha}(x)\}$, and we claim it is the least upper bound.
To see this, assume $F_{u_\alpha}(x) \le y$ for all α. Hence, $x \wedge nu_\alpha \le y$ holds for all α, n. Consequently, $u_\alpha \uparrow u$ implies $x \wedge nu \le y$ for all n, and therefore $F_u(x) = \sup\{x \wedge nu\} \le y$. Hence, $F_u(x)$ is the least upper bound of $\{F_{u_\alpha}(x)\}$, and thus $F_{u_\alpha}(x) \uparrow F_u(x)$.
Let e be a positive element of a Riesz space X. An element $x \in X^+$ is said to be a **component** of e whenever $x \wedge (e - x) = 0$. The collection of all components of e will be denoted by ℓ_e; $\ell_e = \{x \in X^+ : x \wedge (e - x) = 0\}$
If $x \in \ell_e$ then $e - x \in \ell_e$. Also, $F_V e \in \ell_e$ for each projection band V. Under the partial ordering induced by X, the set of components ℓ_e is a Boolean algebra, consisting precisely of the extreme points of the order interval $[0, e]$. The details follow.
Theorem 10. If e is positive element in Riesz space X , then the set of all components ℓ_e of e is a **Boolean algebra** consisting precisely of all extreme

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points of the convex set $[0,e]$. Moreover, in case X is Dedekind complete, ℓ_e is likewise Dedekind complete.

Proof. To see that ℓ_e is a Boolean algebra, let $x,y\in\ell_e$. Then it follows from

$$(x\vee y)\wedge(e-x\vee y)$$
$$=(x\vee y)\wedge[(e-x)\wedge(e-y)]$$
$$=[x\wedge(e-x)\wedge(e-y)]$$
$$\vee[y\wedge(e-x)\wedge(e-y)]=0\vee 0=0$$ and
$$(x\vee y)\wedge(e-x\vee y)$$
$$=(x\wedge y)\wedge[(e-x)\vee(e-y)]$$
$$=[x\wedge y\wedge(e-x)]\vee$$
$$[x\wedge y\wedge(e-y)]=0\vee 0=0$$

That $x\vee y$ and $x\wedge y$ both belong to ℓ_e. Therefore, ℓ_e is a Boolean algebra under the partial ordering induced by X.

Now suppose that X is Dedekind complete. Let U be a nonempty subset of ℓ_e. Then $x=\sup U$ exists in X. Since $0\le y\wedge(e-x)]\le y\wedge(e-y)]=0$ holds for all $y\in U$, it follows:

$x\wedge(e-x)=\sup\{y\wedge(e-x):y\in U\}=0$, and so $x\in\ell_e$. Also, x is the least upper bound of U in ℓ_e.

Now, we shall show that an element is a component of e if and only if it is an extreme point of $[0,e]$. To proof that, let $x\in X^+$.

Assume first that $x\in\ell_e$. Let $x=\lambda y+(1-\lambda)z$ with $y,z\in[0,e]$ and $0<\lambda<1$. From $x\wedge(e-x)=0$ we have $y\wedge(e-x)=0$,

and so

$y=y\wedge e=y\wedge x+y\wedge(e-x)=y\wedge x\le x$

holds, similarly, $z\le x$. Now if either $y<x$ or $z<x$ is true, then

$x=\lambda y+(1-\lambda)z<\lambda x+(1-\lambda)x=x$

Is also true, which is a contradiction. Hence, $y=z=x$ holds, and so x is an extreme point of ℓ_e.

Conversely, let x be an extreme point of $[0,e]$. We have to show that $x\wedge(e-x)=0$. Therefore, let $u=x\wedge(e-x)$. Clearly, $0\le x-u\le e$ and

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$0 \le x+u \le e$, and from the convex combination $x = \frac{1}{2}(x-u) + \frac{1}{2}(x+u)$ we get $u = 0$. Therefore, $x \in \ell_e$, and the proof is finished.

When X has the principal projection property, (Abramovic, 1971, 511) has described the lattice operations of $\ell_b(E,F)$ in terms of components as follows.

Theorem 11. (Abramovic). Assume that X has the principal projection property and that Y is Dedekind complete**.** Then for each $S,T \in \ell_b(X,Y)$ we have

$$S \vee T(x) = \sup\{S(y)+T(z) : y \wedge z = 0 \ \ and \ \ y+z = x\}$$

And

$$S \wedge T(x) = \inf\{S(y)+T(z) : y \wedge z = 0 \ \ and \ \ y+z = x\}$$

For each $x \in X^+$.

Proof. The first formula follows from the second. Indeed, if the second is true, then

$$S \vee T(x) = -(-S) \wedge (-T)(x)$$
$$= -\inf\{-S(y) - T(z) : y \wedge z = 0 \ \ and \ \ y+z = x\}$$
$$= \sup\{S(y)+T(z) : y \wedge z = 0 \ \ and \ \ y+z = x\}$$

Also, if the second formula holds for the special case $S \wedge T = 0$, then it is true in general. Indeed, if this is the case, then the identity

$$(S - S \wedge T) \wedge (T - S \wedge T) = 0$$

Implies that

$$\inf\{S(y)+T(z) : y \wedge z = 0 \ \ and \ \ y+z = x\} - S \wedge T(x)$$

$$= \inf\{[S(y) - S \wedge T(y)] + [T(z) - S \wedge T(z)] : y \wedge z = 0 \ \ and \ \ y+z = x\}$$
$$= (S - S \wedge T) \wedge (T - S \wedge T)(x) = 0.$$

And so

$$S \wedge T(x) = \inf\{S(y)+T(z) : y \wedge z = 0 \ \ and \ \ y+z = x\}.$$

To complete the proof, assume that $S \wedge T = 0$ in $\ell_b(X,Y)$. Fix $x \in X^+$, and let $y \in X^+$ satisfy $0 \le y \le x$. Denote by P the order projection of X onto the band generated by $(2y-x)^+$, and put $z = Px$. From $x \le 2y + (x-2y)^+$ and $(x-2y)^+ \wedge (2y-x)^+ = 0$, it follows that $Px \le 2Py + P(x-2y)^+ = 2Py \le 2y$

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. That is,

$$z \le 2y \qquad (*)$$

Holds. Also, from $(2y-x)^+ \le (2x-x)^+ = x$ we see that

$2y - x \le (2y-x)^+ = P(2y-x)^+ \le Px = z$, and so

$$x - z \le 2(x-y). \qquad (**)$$

Combining (*) and (**), we get

$$S(z) + T(x-z) \le 2[S(y) + T(x-y)],$$

from which it follows that

$$\inf\{S(z) + T(x-z) : z \wedge (x-z) = 0\} = 0.$$

The proof of the theorem is finished.

It should be noted that theorem 11 is false without assuming that X has the principal projection property. For instance, let $X = C[0,1]$, $Y = R$, and let $S, T : X \to Y$ be defined by $S(f) = f(0)$, and $T(f) = f(1)$. Then $S \wedge T = 0$ holds, while

$\inf\{S(f) + T(g) : f \wedge g = 0 \, and \, f + g = 1\}$

$= \inf\{S(f) + T(1-f) : f = 0 \, or \, f = 1\} = 1.$

When X has the principal projection property, the lattice operations of $\ell_b(X,Y)$ can also be expressed in terms of directed systems involving components as follows.

Theorem 12. Assume that X has the principal projection property and that Y is Dedekind complete. Then for all $S, T \in \ell_b(X,Y)$ and $x \in X^+$ we have

1. $\{\sum_{i=1}^{n} S(x_i) \vee T(x_i) : x_i \wedge x_j = 0 \; for \, i \ne j \; and \sum_{i=1}^{n} x_i = x\} \uparrow S \vee T(x);$
2. $\{\sum_{i=1}^{n} S(x_i) \vee T(x_i) : x_i \wedge x_j = 0 \; for \, i \ne j \; and \sum_{i=1}^{n} x_i = x\} \downarrow S \wedge T(x);$
3. $\{\sum_{i=1}^{n} |T(x_i)| : x_i \wedge x_j = 0 \; for \, i \ne j \; and \sum_{i=1}^{n} x_i = x\} \uparrow |T|(x).$

Proof. Since (2) and (3) follow from (1) by using the identities $S \wedge T = -(-S) \vee (-T)$ and $|T| = T \vee (-T)$, we prove only the first formula.

Put

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$D = \{\sum_{i=1}^{n} S(x_i) \vee T(x_i) : x_i \wedge x_j = 0 \; for \; i \neq j \; and \sum_{i=1}^{n} x_i = x\}$, where $x \in X^+$ is fixed, and note that $D \leq S \vee T(x)$ holds in Y. On the other hand, if $y, z \in X^+$ satisfy $y \wedge z = 0$ and $y + z = x$, then the relation
$S(y) + T(z) \leq S(y) \vee T(y) + S(z) \vee T(z) \in D$,
Coupled with Theorem 11, shows that $\sup D = S \vee T(x)$ holds. Therefore, what remains to be shown is that D is directed upward.
To this end, let $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_m\}$ be two subsets of E^+ such that $x_i \wedge x_j = 0$ for $i \neq j$ and $y_p \wedge y_q = 0$ for $p \neq q$ and with $\sum_{i=1}^{n} x_i = \sum_{j=1}^{m} y_j = x$.
Then the finite set $\{x_i \wedge y_j : 1 \leq i \leq n; 1 \leq j \leq m\}$ is pairwise disjoint such that

$$\sum_{i=1}^{n}\sum_{j=1}^{m} x_i \wedge y_j = \sum_{i=1}^{n} x_i \wedge \left(\sum_{j=1}^{m} y_j\right) = \sum_{i=1}^{n} x_i \wedge x = \sum_{i=1}^{n} x_i = x.$$

In addition, we have

$$\begin{aligned}\sum_{i=1}^{n} S(x_i) \vee T(x_i) &= \sum_{i=1}^{n} S\left(x_i \wedge \sum_{j=1}^{m} y_j\right) \vee T\left(x_i \wedge \sum_{j=1}^{m} y_j\right) \\ &= \sum_{i=1}^{n}\left[\sum_{j=1}^{m} S(x_i \wedge y_j)\right] \vee \left[\sum_{j=1}^{m} T(x_i \wedge y_j)\right] \\ &\leq \sum_{i=1}^{n}\sum_{j=1}^{m} S(x_i \wedge y_j) \vee T(x_i \wedge y_j) \text{ , and,}\end{aligned}$$

similarly,

$$\sum_{j=1}^{m} S(y_j) \vee T(y_j) \leq \sum_{i=1}^{n}\sum_{j=1}^{m} S(x_i \wedge y_j) \vee T(x_i \wedge y_j).$$

Therefore, $D \uparrow S \vee T(x)$ holds.
Now we will show important statement of a retracts of Riesz spaces. Let us say that a Riesz subspace G of a Riesz space X is a **retract** of X (or that X is **retractable** on G) whenever there exists a positive projection $F : X \to Y$ (such that $0 \leq F = F^2$) having G as its range.
Theorem 13. For a Riesz subspace G of a Riesz space X the following statements hold:

1. If G a retract of X is Dedekind complete, then G is Dedekind complete Riesz space in its own right.
2. If G is Dedekind complete in its own right and G majorizes X, then G is a retract of X.

Proof. (1) Let $F: X \to Y$ be a positive projection whose range is G, and let $0 \le x_\alpha \uparrow \le x$ in G. Then there exists some $y \in X$ with $0 \le x_\alpha \uparrow y \le x$ in X, and so $0 \le x_\alpha = Fx_\alpha \le Fy$ holds in G for each α. On the other hand, if for some $z \in G$ we have $0 \le x_\alpha \le z$ for all α, then $y \le z$, and hence $Fy \le Fz = z$. In other words, $0 \le x_\alpha \uparrow Fy$ holds in G, which proves that G is a Dedekind complete Riesz space.
(2) Apply Theorem (Kantorvic), which is proof that every positive operator whose domain is a majorizing vector subspace and whose values are in a Dedekind complete Riesz space always has a positive extension to the identity operator $I: G \to G$.[3]

Conclusion:
In the end of this paper we conclude that, a Riesz subspace G of Riesz space X hold that, if G a retract of X is Dedekind complete, then G is Dedekind complete Riesz space in its own right, also if G is Dedekind complete in its own right and G majorizes always has positive extension to the identity positive projection operator $I: G \to G$.

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